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Existence of Triple Positive Solutions for Nonlinear Second Order Arbitrary Two-point Boundary Value Problems

Asaduzzaman, M. *[1] and Ali, M. Z.[2]

[1] *Department of Mathematics, Islamic University, Kushtia-7003, Bangladesh*
[2] *Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh*

E-mail: asad@math.iu.ac.bd
* *Corresponding author*

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ABSTRACT

In this paper, we establish the criteria for existence of triple positive solutions to the nonlinear second order ordinary differential equation $u''(t) + f(t, u(t), u'(t)) = 0$, $t \in [a, b]$, with the arbitrary two-point boundary value conditions $u(a) = u(b) = 0$, where, a, b are two arbitrary non-negative constants and $f \in C([a, b] \times [0, \infty) \times \mathbf{R}, [0, \infty))$. The analysis of this paper is based on a fixed point theorem of functional type in a cone due to Bai and Ge. The result of this paper generalizes the results of several authors in literature. Finally, we give an illustrative example to support our analytic proof.

Keywords: Nonlinear second order arbitrary two-point boundary value problem, Triple positive solutions, Fixed point theorem.

1. Introduction

Literature may contain a huge number of applications of boundary value problems for ordinary differential equations and different kinds of physical, biological and chemical phenomena has been explained by these boundary value problems. For instance, we may revise the works of Love (1944), Prescott (1961), and Timoshenko and Gere (1961) on elasticity, the monographs by Mansfield (1964) and Soedel (1993) on deformation of structures and the work of Dulacska (1992) on the effects of soil. In the last few decades, the existence of positive solutions of two-point, three-point and four-point boundary value problems for second order nonlinear ordinary differential equations has extensively been studied by using various techniques, see for instance the works of Agarwal and O'Regan (2005), Agarwal et al. (1999), Bai and Du (2007), Bai and Ge (2004), Bai et al. (2004), Guezane-Lakoud and Kelaiaia (2010), Guo and Lakshmlkantham (1988), Henderson and Wang (1997), Ji (2017), Leggett and Williams (1979), Lian et al. (1996), Sun et al. (2009) and Krasnosel'skii (1964). Gue-Krasnosel'skii fixed point theorem given in Guo and Lakshmlkantham (1988), Krasnosel'skii (1964) and Leggett-Williams fixed point theorem of Leggett and Williams (1979) has widely been used to establish the existence criteria of positive solutions for second order ordinary differential equation with different point boundary value problems, see for instance the monographs of Agarwal and O'Regan (2005), Bai and Du (2007), Bai and Ge (2004) and Bai et al. (2004). Using the Leggett-Williams fixed point theorem of Leggett and Williams (1979), Agarwal et al. (1999) established the principle for the existence of three positive solutions to a class of second order impulsive differential equations.

From the works of Avery (1998),Avery and Henderson (2000), Avery and Henderson (2001), Anderson and Avery (2002) and Avery and Peterson (2001) it is clear that five functional fixed point theorem given by Avery (1998), Avery and Henderson (2000), fixed point theorem of cone expansion and compression of functional type given by Avery and Henderson (2001), twin fixed point theorem given by Anderson and Avery (2002) and generalized Leggett-Williams fixed point theorem given by Avery and Peterson (2001) all are extension of Gue-Krasnosel'skii fixed point theorem given in Guo and Lakshmlkantham (1988), Krasnosel'skii (1964) and Leggett-Williams fixed point theorem given in Leggett and Williams (1979). In 2004, Bai and Ge (2004) established a new fixed point theorem (Theorem 2.1 of Bai and Ge (2004)) by generalizing Leggett-Williams fixed point theorem given in Leggett and Williams (1979) and using this new fixed point theorem (Theorem 2.1 of Bai and Ge (2004)) they established some new multiplicity results for the following nonlinear second order two-point boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{1}$$

where, $f \in C\left([0,\, 1] \times [0,\, \infty) \times \mathbf{R},\, [0,\, \infty)\right)$.

To the best of our knowledge there is no any work on the existence of positive solutions for nonlinear second order boundary value problem with arbitrary point boundary value conditions. From this context, in this paper we establish the criteria for existence of three positive solutions to the following nonlinear second order ordinary differential equation:

$$u''(t) + f(t,\, u(t),\, u'(t)) = 0,\;\; t \in [a,\, b], \tag{2}$$

under the following arbitrary two-point boundary value conditions:

$$u(a) = u(b) = 0, \tag{3}$$

where, a, b are two arbitrary non-negative constants and

$$f \in C\left([a,\, b] \times [0,\, \infty) \times \mathbf{R},\, [0,\, \infty)\right),$$

applying the fixed point theorem due to Bai and Ge (2004). The rest of this paper is furnished as follows:
In Section 2, we provide some basic definitions, a lemma and the fixed point theorem due to Bai and Ge (2004). In Section 3, we state and prove our main results, which provide us the techniques to check the existence of three positive solutions of second order arbitrary non-negative two-point boundary value problem (2) and (3) under some certain assumptions. In Section 4, we give an example which helps us to illustrate our main result.

2. Preliminary Notes

In this section we recall some basic definitions, the fixed point theorem in a cone due to Bai and Ge (2004) and establish a lemma which are needed to prove our main results.

Definition 2.1: Let $(B, \|.\|)$ be a real Banach space and P be a nonempty closed convex subset of B. Then we say that P is a cone on B if it is satisfies the following properties:
(i) $\eta c \in P$ for $c \in P$, $\eta \geq 0$;

(ii) c, $c \in P$ implies $c = \theta$,
where θ denotes the null element of B.

Definition 2.2: A mapping γ is said to be a non-negative continuous concave functional on the cone P if $\gamma : P \to [0, +\infty)$ is continuous and

$$\gamma(\delta x + (1-\delta)y) \geq \delta\gamma(x) + (1-\delta)\gamma(y),$$

for all x, $y \in P$, $\delta \in [0, 1]$.

Definition 2.3: A mapping α is said to be a non-negative continuous convex functional on the cone P if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(\delta x + (1-\delta)y) \leq \delta\alpha(x) + (1-\delta)\alpha(y),$$

for all x, $y \in P$, $\delta \in [0, 1]$.

Definition 2.4: Suppose α, $\beta : P \to [0, +\infty)$ are two non-negative continuous convex functionals satisfying

$$\|u\| \leq M max\left\{\alpha(u),\, \beta(u)\right\},\ for\, each\, u \in P, \tag{4}$$

where M is a positive constant, and

$$\Omega = \{u \in P : \alpha(u) < r,\, \beta(u) < L\} \neq \Phi,\ for\, r > 0,\, L > 0. \tag{5}$$

From (4) and (5), we have Ω is a bounded nonempty open subset in P .

Definition 2.5: Let $r > a_1 > 0$, $L > 0$ be given, α, $\beta : P \to [0, +\infty)$ are two non-negative continuous convex functionals satisfying (4) and (5), and α be a non-negative continuous concave functional on the cone P. Define the following bounded convex sets:

$$P(\alpha, r; \beta, L) = \left\{u \in P : \alpha(u) < r,\, \beta(u) < L\right\},$$
$$\bar{P}(\alpha, r; \beta, L) = \left\{u \in P : \alpha(u) \leq r,\, \beta(u) \leq L\right\},$$
$$P(\alpha, r; \beta, L; \gamma, a_1) = \left\{u \in P : \alpha(u) < r,\, \beta(u) < L,\, \gamma(u) > a_1\right\},$$
$$\bar{P}(\alpha, r; \beta, L; \gamma, a_1) = \left\{u \in P : \alpha(u) \leq r,\, \beta(u) \leq L,\, \gamma(u) \geq a_1\right\}.$$

Now, we state a fixed point theorem on the cone P due to Bai and Ge (2004).

Theorem 2.1. *(Theorem 2.1 of Bai and Ge (2004)) Let B be a Banach space, $P \subset B$ be a cone and $r_2 \geq c_1 > b_1 > r_1 > 0$, $L_2 \geq L_1 > 0$ be given. Assume that α, β are two non-negative continuous convex functionals on P, such that*

(4) and (5) are satisfied, γ is a non-negative continuous concave functional on P, such that $\gamma(u) \leq \alpha(u)$, for all $u \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2)$ and let

$$A : \bar{P}(\alpha,\, r_2;\, \beta,\, L_2) \to \bar{P}(\alpha,\, r_2;\, \beta,\, L_2)$$

be a completely continuous operator. Suppose

(C_1) $\quad \left\{u \in \bar{P}(\alpha,\, c_1;\, \beta,\, L_2;\, \gamma,\, b_1) : \gamma(u) > b_1\right\} \neq \Phi,\ \gamma(Au) > b_1,$ *for* $u \in \bar{P}(\alpha,\, c_1;\, \beta,\, L_2;\, \gamma,\, b_1)$,

(C_2) $\quad \alpha(Au) < r_1,\ \beta(Au) < L_1,$ *for all* $u \in \bar{P}(\alpha,\, r_1;\, \beta,\, L_1)$,

(C_3) $\quad \gamma(Au) > b_1,$ *for all* $u \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1)$ *with* $\alpha(Au) > c_1$.

Then A has at least three fixed points u_1, u_2 and u_3 in $\bar{P}(\alpha,\, r_2;\, \beta,\, L_2)$. Further,

$$u_1 \in P(\alpha,\, r_1;\, \beta,\, L_1),\ \ u_2 \in \left\{\bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1) : \gamma(u) > b_1\right\},$$

and

$$u_3 \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2) \setminus \left(\bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1) \cup \bar{P}(\alpha,\, r_1;\, \beta,\, L_1)\right).$$

Definition 2.6: A solution $u(t)$ of the boundary value problem given by (2) and (3) is said to be a positive solution if $u(t) > 0$ for all $t \in (a,\, b)$.

Lemma 2.1. *Assume that $0 \leq a < b$. If $h(t) \in C[a,\, b]$, for all $t \in [a,\, b]$, then the unique solution of following nonlinear second order arbitrary two-point boundary value problem*

$$\begin{cases} -u''(t) = h(t),\ t \in [a,\, b], \\ u(a) = u(b) = 0, \end{cases} \tag{6}$$

is $u(t) = \int_a^b G(t,\, s)h(s)ds$, where, $G(t,\, s)$ is the Green's function of the corresponding homogeneous second order arbitrary two-point boundary value problem

$$\begin{cases} -u''(t) = 0,\ t \in [a,\, b], \\ u(a) = u(b) = 0, \end{cases} \tag{7}$$

that is,

$$G(t,\, s) = \frac{1}{(b-a)} \begin{cases} (t-a)(b-s);\ \ a \leq t \leq s \leq b, \\ (s-a)(b-t);\ \ a \leq s \leq t \leq b. \end{cases} \tag{8}$$

Definition 2.7: Let $B = C[a,\, b]$ be processed with the ordering $u \leq v$ if $u(t) \leq v(t)$, for all $t \in [a,\, b]$, and the maximum norm,

$$\|u\| = max \left\{max_{a \leq t \leq b} \mid u(t) \mid,\, max_{a \leq t \leq b} \mid u'(t) \mid\right\}.$$

From the fact that $u''(t) = -f(t, u(t), u'(t)) \leq 0$, we obtain that u is concave on $[a, b]$. Thus, we define a cone $P \subset B$ by

$$P = \{u \in C[a, b] : u(t) \geq 0, \, u\, is\, concave\, on\, [a, b], \, t \in [a, b]\} \subset B. \tag{9}$$

Furthermore, for $u \in P$ if we define the functionals

$$\alpha(u) = max_{a \leq t \leq b} \mid u(t) \mid, \, \beta(u) = max_{a \leq t \leq b} \mid u'(t) \mid,$$
$$\gamma(u) = min_{\frac{3a+b}{4} \leq t \leq \frac{a+3b}{4}} \mid u(t) \mid,$$

then $\alpha, \beta, \gamma : P \to [0, +\infty)$ are three continuous non-negative functionals such that $\|u\| = max\,\{\alpha(u), \beta(u)\}$, and (4) and (5) hold; α, β are convex, γ is concave and $\gamma(u) \leq \alpha(u)$ holds for all $u \in P$.

Remark 2.1. *By Lemma 2.1, we can convert the boundary value problem given by (2) and (3) as in the following integral equation*

$$u(t) = \int_b^a G(t, s) f(s, u(s), u'(s)) ds, \; for\, all\, t \in [a, b], \tag{10}$$

where $G(t, s)$ is the Green's function given by (8). It is also noted that, the Green's function $G(t, s)$ have the following properties:
(i) $G(t, s)$ *is continuous on* $[a, b] \times [a, b]$,
(ii) $G(a, s) = G(b, s) = G'(a, s) = G'(b, s)$, *for all* $s \in [a, b]$ *and*
(ii) $G(t, s) \geq 0$, *for all* $t, s \in [a, b]$.

Obviously, $u = u(t)$, for all $t \in [a, b]$ is a solution of the boundary value problem given by (2) and (3), if and only if it is a solution of the integral equation (10). Furthermore, if we consider a cone P on $C[a, b]$and define an integral operator $A : P \to P$ by

$$Au(t) = \int_b^a G(t, s) f(s, u(s), u'(s)) ds, \; for\, all\, u \in P, \tag{11}$$

then it is easy to see that the boundary value problem given by (2) and (3) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A defined by (11).

3. Main Results

In this section, we present and prove our main results.

Throughout this paper, we suppose that

$$\lambda = min \left\{ \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G\left(\frac{3a+b}{4}, s\right) ds, \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G\left(\frac{a+3b}{4}, s\right) ds \right\} = \frac{(b-a)^3}{16}.$$

Theorem 3.1. *If there exist some constants* $r_2 \geq \frac{4}{3a+b} \cdot b_1 > b_1 > r_1 > 0$, $L_2 \geq L_1 > 0$ *such that* $\frac{b_1}{\lambda} \leq min\left\{\frac{8}{(b-a)^2} \cdot r_2,\, \frac{2}{(b-a)^2} \cdot L_2\right\}$ *and the following conditions are satisfied:*

$$(H_1)\, f(t,\, u,\, v) < min\left\{\frac{8}{(b-a)^2} \cdot r_1,\, \frac{2}{(b-a)^2} \cdot L_1\right\},$$
$$for\, (t,\, u,\, v) \in [a,\, b] \times [a,\, r_1] \times [-L_1,\, L_1];$$
$$(H_2)\, f(t,\, u,\, v) > \frac{b_1}{\lambda},$$
$$for\, (t,\, u,\, v) \in \left[\frac{3a+b}{4},\, \frac{a+3b}{4}\right] \times \left[b_1,\, \frac{4}{3a+b} \cdot b_1\right] \times [-L_2,\, L_2];$$
$$(H_3)\, f(t,\, u,\, v) \leq min\left\{\frac{8}{(b-a)^2} \cdot r_2,\, \frac{2}{(b-a)^2} \cdot L_2\right\},$$
$$for\, (t,\, u,\, v) \in [a,\, b] \times [a,\, r_2] \times [-L_2,\, L_2],$$

then the boundary value problem given by (2) and (3) has at least three positive solutions, u_1, u_2 *and* u_3 *satisfying*

$$max_{a \leq t \leq b} u_1(t) \leq r_1,\ max_{a \leq t \leq b} \|u_1'(t)\| \leq L_1;$$
$$b_1 < max_{\frac{3a+b}{4} \leq t \leq \frac{a+3b}{4}} u_2(t) \leq max_{a \leq t \leq b} u_2(t) \leq r_2,\ max_{a \leq t \leq b} \|u_1'(t)\| \leq L_2;$$
$$max_{a \leq t \leq b} u_3(t) \leq \frac{4}{3a+b} \cdot b_1,\ max_{a \leq t \leq b} \|u_3'(t)\| \leq L_2.$$

Proof. First, we define the integral operator $A : P \to P$ by

$$u(t) = Au(t) := \int_b^a G(t,\, s) f(s,\, u(s),\, u'(s)) ds,\ for\, all\, t \in [a,\, b],\, u \in C[a,\, b]. \tag{12}$$

Then, according to the *Remark 2.1*, the boundary value problem given by (2) and (3) has a solution $u = u(t)$ if and only if $u = u(t)$ is a solution of the integral equation (12).

By Arzela-Ascoli theorem given in Frechet (1906), it is obvious that $A : P \to P$ is completely continuous. Now, we will prove that all the conditions of *Theorem 2.1* (fixed point theorem on cone due to Bai and Ge (2004)) satisfy.

If $u \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2)$, then we have $\alpha(u) \leq r_2$, $\beta(u) \leq L_2$ and the condition (H_3) gives us

$$f(t,\, u(t),\, u'(t)) \leq min\left\{\frac{8}{(b-a)^2} \cdot r_2,\, \frac{2}{(b-a)^2} \cdot L_2\right\}.$$

Accordingly,

$$\begin{aligned}\alpha(Au) &= max_{a\le t\le b} \mid \int_b^a G(t,\, s)f(s,\, u(s),\, u'(s))ds \mid \\ &\le \frac{8}{(b-a)^2}\cdot r_2 \cdot max_{a\le t\le b}\int_b^a G(t,\, s)ds \\ &= \frac{8}{(b-a)^2}\cdot r_2\cdot \frac{(b-a)^2}{8} = r_2.\end{aligned}$$

Now, for $u \in P$, we have $Au \in P$. Thus Au is concave on $[a,\, b]$, and

$$max_{a\le t\le b} \mid (Au)'(t) \mid = max\left\{\mid (Au)'(a) \mid,\, \mid (Au)'(b) \mid\right\}.$$

Consequently,

$$\begin{aligned}\beta(Au) &= max_{a\le t\le b} \mid (Au)'(t) \mid \\ &= max_{a\le t\le b} \mid -\int_a^t (s-a)f(s,\, u(s),\, u'(s))ds \\ &\quad + \int_t^b (b-s)f(s,\, u(s),\, u'(s))ds \mid \\ &= max\left\{\int_a^b (b-s)f(s,\, u(s),\, u'(s))ds,\, \int_a^b (s-a)f(s,\, u(s),\, u'(s))ds\right\} \\ &\le \frac{2}{(b-a)^2}\cdot L_2\cdot max\left\{\int_a^b (b-s)ds,\, \int_a^b (s-a)ds\right\} \\ &= \frac{2}{(b-a)^2}\cdot L_2\cdot\frac{(b-a)^2}{2} = L_2.\end{aligned}$$

Hence,A maps $\bar{P}\,(\alpha,\, r_2;\, \beta,\, L_2)$ into itself. Similarly, if we consider $u \in \bar{P}(\alpha,\, r_1;\, \beta,\, L_1)$, then we have $\alpha(u) \le r_1$, $\beta(u) \le L_1$ and the condition (H_1) gives us

$$f(t,\, u(t),\, u'(t)) \le min\left\{\frac{8}{(b-a)^2}\cdot r_1,\, \frac{2}{(b-a)^2}\cdot L_1\right\},\, for\, t\in[a,\, b],$$

and if we maintain the above procedure, then it is obvious that A maps $\bar{P}\,(\alpha,\, r_1;\, \beta,\, L_1)$ into itself. Therefore, the condition (C_2) of *Theorem 2.1* is satisfied.

To satisfy the condition (C_1) of *Theorem 2.1*, we take $u(t) = \frac{4}{3a+b} \cdot b_1$, for $t \in [a, b]$. It is easy to see that

$$u(t) = \frac{4}{3a+b} \cdot b_1 \in \bar{P}\left(\alpha,\, \frac{4}{3a+b} \cdot b_1;\, \beta,\, L_2; \gamma,\, b_1\right)$$

and

$$\gamma(u) = \gamma\left(\frac{4}{3a+b} \cdot b_1\right) > b_1.$$

So, $\left\{u \in \bar{P}\left(\alpha,\, \frac{4}{3a+b} \cdot b_1;\, \beta,\, L_2; \gamma,\, b_1\right) : \gamma(u) > b_1\right\} \neq \Phi$.

Thus, if $u(t) \in \bar{P}\left(\alpha,\, \frac{4}{3a+b} \cdot b_1;\, \beta,\, L_2; \gamma,\, b_1\right)$, then

$$b_1 \leq u(t) \leq \frac{4}{3a+b} \cdot b_1,\ for\ \frac{3a+b}{4} \leq t \leq \frac{a+3b}{4}.$$

Now, from condition (H_2), we have $f(t,\, u(t),\, u'(t)) > \frac{b_1}{\lambda}$, for $\frac{3a+b}{4} \leq t \leq \frac{a+3b}{4}$ and by the definitions of γ and the cone P, we obtain following two cases:
(I) $\gamma(Au) = (Au)\left(\frac{3a+b}{4}\right)$ and
(II) $\gamma(Au) = (Au)\left(\frac{a+3b}{4}\right)$.

In case (I), we have

$$\begin{aligned}
\gamma(Au) &= \int_a^b G\left(\frac{3a+b}{4},\, s\right) f(s,\, u(s),\, u'(s))ds \\
&> \frac{b_1}{\lambda} \cdot \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G\left(\frac{3a+b}{4},\, s\right) ds \\
&\geq \frac{b_1}{\lambda} \cdot \frac{(b-a)^3}{16} = b_1.
\end{aligned}$$

In case (II), we have

$$\begin{aligned}
\gamma(Au) &= \int_a^b G\left(\frac{a+3b}{4},\, s\right) f(s,\, u(s),\, u'(s))ds \\
&> \frac{b_1}{\lambda} \cdot \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} G\left(\frac{a+3b}{4},\, s\right) ds \\
&\geq \frac{b_1}{\lambda} \cdot \frac{(b-a)^3}{16} = b_1.
\end{aligned}$$

That is $\gamma(Au) > b_1$, for all $u \in \bar{P}(\alpha,\, 4b_1;\, \beta,\, L_2;\, \gamma,\, b_1)$. This proves that the condition (C_1) of *Theorem 2.1* is satisfied.

Finally, we prove that the condition (C_3) of *Theorem 2.1* also satisfies. If we consider $u \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1)$ with $\alpha(Au) > \frac{4}{3a+b} \cdot b_1$, then by the definition of γ and for $Au \in P$, we have

$$\begin{aligned}
\gamma(Au) &= min_{\frac{3a+b}{4} \le t \le \frac{a+3b}{4}} (Au)(t) \\
&\ge \frac{3a+b}{4} \cdot max_{a \le t \le b} (Au)(t) \\
&= \frac{3a+b}{4} \cdot \alpha(Au) \\
&> \frac{3a+b}{4} \cdot \frac{4}{3a+b} \cdot b_1 = b_1.
\end{aligned}$$

Thus, the condition (C_3) of *Theorem 2.1* is satisfied. Hence, all conditions of *Theorem 2.1* are hold for the integral operator A defined by (11). Therefore, according to the *Theorem 2.1*, we can say that the integral operator A has at least three fixed points u_1, u_2 and u_3 in $\bar{P}\,(\alpha,\, r_2;\, \beta,\, L_2)$ satisfying

$$\begin{cases} u_1 \in P(\alpha,\, r_1;\, \beta,\, L_1), \;\; u_2 \in \left\{ \bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1) : \gamma(u) > b_1 \right\}, \\ and \\ u_3 \in \bar{P}(\alpha,\, r_2;\, \beta,\, L_2) \setminus \left(\bar{P}(\alpha,\, r_2;\, \beta,\, L_2;\, \gamma,\, b_1) \cup \bar{P}(\alpha,\, r_1;\, \beta,\, L_1) \right). \end{cases} \tag{13}$$

In addition, since u_3 satisfies $\alpha(u_3) \le \frac{4}{3a+b} \cdot \gamma(u_3)$, then

$$max_{a \le t \le b} u_3(t) \le \frac{4}{3a+b} \cdot b_1.$$

Hence, *Lemma 2.1*, *Remark 2.1* and (13) confirm that the boundary value problem given by (2) and (3) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\begin{aligned}
&max_{a \le t \le b} u_1(t) \le r_1, \;\; max_{a \le t \le b} \|u_1'(t)\| \le L_1; \\
&b_1 < max_{\frac{3a+b}{4} \le t \le \frac{a+3b}{4}} u_2(t) \le max_{a \le t \le b} u_2(t) \le r_2, \;\; max_{a \le t \le b} \|u_1'(t)\| \le L_2; \\
&max_{a \le t \le b} u_3(t) \le \frac{4}{3a+b} \cdot b_1, \;\; max_{a \le t \le b} \|u_3'(t)\| \le L_2.
\end{aligned}$$

This completes the proof. $\square$

Furthermore, using *Theorem 2.1*, we obtain that

$$max_{a\le t\le b}u_3(t)\le r_2,\ min_{\frac{3a+b}{4}\le t\le\frac{a+3b}{4}}u_3(t)<b_1$$

and if for boundary value problem given by (2) and (3), the functionals α and γ satisfy the following additional relation:

$$\begin{aligned}\gamma(u)&=min_{\frac{3a+b}{4}\le t\le\frac{a+3b}{4}}u(t)\\&\ge\frac{3a+b}{4}\cdot max_{a\le t\le b}u(t)\\&=\frac{3a+b}{4}\cdot\alpha(u),\ for\,u\in P,\end{aligned}$$

then, we yield that $max_{a\le t\le b}u_3(t)\le\frac{4}{3a+b}\cdot b_1$.

In the above mentioned case *Theorem 3.1* leads the following corollary:

Corollary 3.1. *If there exist some constants*

$$\begin{aligned}&0<r_1<b_2\le\frac{4}{3a+b}\cdot b_2\le r_2<b_3<\frac{4}{3a+b}\cdot b_3\le\cdots\le r_n,\\&0<L_1\le L_2\le L_3\le\cdots\le L_{n-1},\ n\in\boldsymbol{N},\end{aligned}$$

such that $\frac{b_{i+1}}{\lambda}\le min\left\{\frac{8}{(b-a)^2}\cdot r_{i+1},\ \frac{2}{(b-a)^2}L_{i+1}\right\}$ *and the following conditions are satisfied:*

$$\begin{aligned}&(H_4)\,f(t,\,u,\,v)<min\left\{\frac{8}{(b-a)^2}\cdot r_i,\ \frac{2}{(b-a)^2}\cdot L_i\right\},\\&\qquad for\,(t,\,u,\,v)\in[a,\,b]\times[a,\,r_i]\times[-L_i,\,L_i],\ 1\le i\le n;\\&(H_5)\,f(t,\,u,\,v)>\frac{b_{i+1}}{\lambda},\ for\,(t,\,u,\,v)\in\left[\frac{3a+b}{4},\ \frac{a+3b}{4}\right]\\&\qquad\times\left[b_{i+1},\ \frac{4}{3a+b}\cdot b_{i+1}\right]\times[-L_{i+1},\,L_{i+1}],\ 1\le i\le n-1,\end{aligned}$$

then the boundary value problem given by (2) and (3) has at least $2n-1$ *positive solutions.*

Proof. We prove this corollary by using the Principle of mathematical induction.

For $n=1$, from condition (H_4) we get

$$A:\bar{P}(\alpha,\,r_1;\,\beta,\,L_1)\to P(\alpha,\,r_1;\,\beta,\,L_1)\subset\bar{P}(\alpha,\,r_1;\,\beta,\,L_1),$$

and hence by Schauder fixed point theorem, we yield at least one fixed point $u_1 \in P(\alpha, r_1; \beta, L_1)$ of A , i.e., the boundary value problem given by (2) and (3) has at least one positive solution.

For $n = 2$, it is clear that the *Theorem 3.1* holds, i.e., the boundary value problem given by (2) and (3) has at least $2 \cdot 2 - 1 = 3$ positive solutions u_2, u_3 and u_4.

Proceeding in this way, if we consider that, for $n = m$ the boundary value problem given by (2) and (3) has at least $2m - 1$ positive solutions, then it is easy to prove that, for $n = m+1$ the boundary value problem given by (2) and (3) has at least $2m + 1$ positive solutions.

This completes the proof. $\square$

Remark 3.1. *Our Theorem 3.1 generalized Theorem 3.1 of Bai and Ge (2004) in the case of arbitrariness of boundary points. It is because we established our theorem under arbitrary two-point boundary value conditions, whereas Bai and Ge (2004) used particular two-point boundary value conditions. Corollary 3.1 shows that the boundary value problem of type (2) and (3) have any number of positive solutions under some additional conditions from our Theorem 3.1.*

4. Applications

In this section, we provide an example to illustrate our main result.

Example 4.1: Consider the following nonlinear second order two-point boundary value problem:

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \ t \in [0, 3], \\ u(0) = u(3) = 0, \end{cases} \tag{14}$$

with $v = u'(t)$ and

$$f(t, u, v) = \begin{cases} \sin t + \frac{9}{2}u^3 + \left(\frac{|v|}{300}\right)^3; & for\, t \in [0, 3],\, u \le 8, \\ \sin t + \frac{9}{2}(9-u)u^3 + \left(\frac{|v|}{300}\right)^3; & for\, t \in [0, 3],\, 8 < u \le 9, \\ \sin t + \frac{9}{2}(u-9)u^3 + \left(\frac{|v|}{300}\right)^3; & for\, t \in [0, 3],\, 9 < u \le 10, \\ \sin t + \frac{4500}{9} + \left(\frac{|v|}{2700}\right)^3; & for\, t \in [0, 3],\, u > 10. \end{cases}$$

Now, if we let $r_1 = 1$, $b_1 = 2$, $r_2 = 1000$, $L_1 = 10$, $L_2 = 3000$, then we get

$$min\left\{\frac{8}{(b-a)^2}\cdot r_1,\ \frac{2}{(b-a)^2}\cdot L_1\right\} = \frac{8}{9},\ \frac{b_1}{\lambda} = \frac{32}{9},$$
$$min\left\{\frac{8}{(b-a)^2}\cdot r_2,\ \frac{2}{(b-a)^2}\cdot L_2\right\} = \frac{6000}{9}$$

and hence

$$\begin{aligned}
f(t,\, u,\, v) &< \frac{8}{9}, && for\, 0 \le t \le 3,\ 0 \le u \le 1,\ -10 \le v \le 10;\\
f(t,\, u,\, v) &> \frac{32}{9}, && for\, \frac{3}{4} \le t \le \frac{9}{4},\ 2 \le u \le \frac{8}{3},\ -3000 \le v \le 3000;\\
f(t,\, u,\, v) &< \frac{6000}{9}, && for\, 0 \le t \le 3,\ 0 \le u \le 1000,\ -3000 \le v \le 3000.
\end{aligned}$$

This means that all the assumptions of *Theorem 3.1* are satisfied. Therefore, according to the *Theorem 3.1*, we can say that the boundary value problem given by (14) has at least three positive solutions u_1, u_2 and u_3 such that

$$\begin{aligned}
&max_{0\le t\le 3} u_1(t) \le 1,\ max_{0\le t\le 3}\|u_1'(t)\| \le 10;\\
&2 < max_{\frac{3}{4}\le t\le \frac{9}{4}} u_2(t) \le max_{0\le t\le 3} u_2(t) \le 1000,\ max_{0\le t\le 3}\|u_1'(t)\| \le 3000;\\
&max_{0\le t\le 3} u_3(t) \le \frac{8}{3},\ max_{0\le t\le 3}\|u_3'(t)\| \le 3000.
\end{aligned}$$

5. Conclusions

In this study, we have established a general criterion for checking the existence of three positive solutions of nonlinear second order arbitrary two-point boundary value problem given by (2) and (3) applying a fixed point theorem due to Bai and Ge (2004). By using *Theorem 3.1*, we can easily checked the existence of three positive solutions to the boundary value problem of type (2) and (3). The result of this paper generalized the corresponding result of Bai and Ge (2004). Our result also generalized the results of Agarwal and O'Regan (2005), Bai and Du (2007) and Agarwal et al. (1999), but they used different fixed point theorems.

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